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Calculating Topological Indices of Benes and Butterfly Network

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Abstract: A graph is numerically represented by topological indices. These indices are crucial for topological indices because they affect the quantity structure property connection and the quantitative structure-activity relationship. In parallel computing, digital signal processing, communications, data centers, and network-on-chip design, benes are utilized. In this article, we calculated the quadratic-Contraharmonic index (QCI), contra harmonic-quadratic index (CQI), geometric quadratic index (GQI), quadratic geometric index (QGI), arithmetic Contraharmonic index (ACI) and Contraharmonic arithmetic index (CAI) for the cylindrical Benes network $HCB(m)$ and for both horizontal and vertical and Butterfly network. We use MATLAB tool to give study these networks graphically and draw the comparison bar graphs.

Keywords: Benes Network; Butterfly Network; Quadratic Geometric Index; Arithmetic Contraharmonic Index; Graphically.

1. Introduction

In an interconnection network, a multiprocessor serves as the processing node to create a network with identical pairings of memory and processors. Programs are put together and executed through message exchanges. The use and architecture of multiprocessor interconnection networks are crucial because of their enhanced affordability and increased efficiency in chips and microprocessors [1]. These networks are valuable and significant because they mimic naturalistic communication patterns. These networks are interdependent and tied to one another, thus further research on them is necessary. Graphs are used to form interconnected networks in a manner that is similar to natural systems.

In these networks, vertices stand in for CPUs or other components, while edges stand in for communication links like fiber optic cables. Incidence functions govern how these elements function. Graphs are used to graphically display the topological aspects of the network. Therefore, networks, their elements, and their connections are fundamentally graphs, their vertices, and their edges. Graphs are essentially used to build networks, with vertices representing processors or other components and edges representing communication connections. The functions of these components are explained by incidence functions, while the topological properties of the network are shown by graphs. Thus, when we talk about edges, vertices, and graphs, we are talking about the building blocks of these networks. In order to facilitate effective Fast Fourier Transform (FFT) operations, butterfly graphs are essential components of Fourier transform networks. These networks use a sequence of linked phases to create connections between input and output pairs. The Benes network, which is made up of butterfly graphs connected by connections, is one of them and is well known for its effectiveness at handling permutation routing. Benes networks are important interconnection networks with very efficient topologies in the field of communication systems. They are used in many parallel computing systems, including as SP1/SP2, IBM, NEC Cenju-3, MIT Transit Project, and SP2/SP2.

In addition, they find application in the interior configurations of optical couplers. In a Benes network with r dimensions, there are 2r + 1 levels altogether in the network, with each level made up of 2r nodes. A butterfly pattern is formed by the connection structure connecting the level 0 and level r nodes. Using the common middle level between two back-to-back butterfly structures, the Benes network is constructed. The standard notation for an r-dimensional Benes network is B(r). As an example, Figure 1 shows a 3-dimensional Benes network. The following notations are used to describe a connected, undirected graph, represented as G: The variables V(G) and E(G) stand for the set of vertices and edges, respectively; da denotes the degree of a vertex 'a', or the number of edges connected to it; N(a) denotes the neighbors of 'a', or the vertices that are directly connected to it, and Sa is the sum of the degrees of the neighboring vertices.

If any terminology is not obvious, the listed references [6–8] provide more clarifications. Different labels applied to graphs or molecular structures reveal information on their composition and how it affects their physical and chemical properties. We call these characteristics topological indices. Several topological indices may be used to predict the physical and chemical characteristics of molecular structures based on the degrees of vertices. These indices give network structures numerical values. The objective is to use these indications to build computer networks and systems that increase productivity and advancement. An interconnection network's structure is determined by how its vertices and edges are arranged, and a graph may be used to visualize this structure. The network's properties are influenced by the graph's arrangement. We may ascertain attributes such as the greatest distance between any two vertices by examining the network's topography.

A vertex's degree indicates the quantity of connections it has. A variety of interconnected networks are studied through the use of graph theory and complex network analysis techniques. These include computer networks (from intranets to global networks), electrical power grids, social networks, robotic networks, transmission networks, and genetic networks. These networks are used for a wide range of reasons and have many applications. The main goal of this study is to find topological invariants (TIs), which are quantitative representations of the structure or patterns of connections between nodes or other elements in a network. These TIs define the parameters of the research topic. A vertex's degree represents how connected it is. Graph theory and sophisticated network analysis techniques are used by researchers to investigate a wide variety of interconnected networks, including computer networks (from intranets to global networks), electrical power grids, social networks, robotics networks, transmission networks, and genetic networks. These networks are rather common and have many uses. The main focus of this work is to find topological invariants (TIs), which are quantitative representations of the structural relationships or connection patterns between nodes or entities in a network. The main goal of this research is to locate these TIs. By understanding these patterns, the research aims to create a comprehensive framework of laws and regulations that govern a variety of natural phenomena. The models developed in this work are supported by quantitative structure-activity relationship (QSAR) and quantitative structure-property relationship (QSPR) approaches. It's important to keep in mind that network analysis with these measures allows different regional shielding design options to be quantitatively evaluated [4].

2. Literature Review

Topological indices (TIs) are widely used to investigate the relationship between the structure and physical characteristics of extremely small materials (nanoscale). The domains of industry, electronics, medical, pharmaceuticals, communication, information technology, and food science can all be significantly impacted by the development of novel nanostructures. The Sombor index and its correlation with entropy, acentric factor, and the vaporization behavior of octane isomers are the main subjects of this study. They investigate this using linear models. The findings indicate a strong correlation between the Sombor index and particular chemical features, particularly DHVAP. In addition, they compute the Sombor index for the various morphologies and configurations of the substance under study.

One area of Mathematical Chemistry that has a big influence on the development of the Chemical Sciences is Chemical Graph Theory. The Combinatorial Quadratic Index (CQI) of molecular graphs is the subject of this research work. The work investigates the CQI for significant nano star dendrimers and a number of typical graph patterns. Along with updated versions of the first and second K-Banhatti Indices of a graph, it also presents the harmonic Sombor indices of a graph. The study looks into freshly created measures. In addition, it looks at TUC4C8[p, q] and TUC4[p, q] nanotubes, the modified first and second Banhatti Sombor indices, and the harmonic Banhatti Sombor index. A variety of topology-related indices based on degrees, distances, and counting techniques are examined in this article. It looks at these indices in relation to various molecular architectures, such as hexagonal cross-sections, honeycomb patterns, and framework networks. Strength, strain energy, boiling temperatures, and other physical and chemical characteristics of molecules having these structures are all represented by these indices. Group theory and graph theory are used in the review to find these traits using Cayley graphs (CG). It computes a number of indices for different sets of graphs, such as the general Randi index, the first Zagreb index, the ABC index, the GA index, the ABC4 index, and the GA5 index. These graphs are the basis for constructing various graph groups related to computers networks and chemicals. The need to understand intricately linked networks is increasing with the introduction of large-scale integrated circuits. Graph theory is essential to the construction and assessment of these networks. It offers a thorough comprehension of these interconnected systems.

A branch of mathematics called chemical graph theory uses the ideas of graph theory to mathematically describe chemical reactions. In the context of networks, edges indicate the channels by which data is sent inside the network, while vertices stand in for network elements such as computers, switches, and other devices. From a scientific standpoint, the development of large-scale integrated circuits has greatly increased our knowledge of complex linked networks, highlighting the critical role that graph theory plays in both their construction and analysis. Graph theory and interconnected networks may be used to gain a thorough knowledge of these interconnected systems. A subfield of mathematics called chemical graph theory uses graph theory methods to explicitly describe chemical reactions. Vertices in the context of networks stand for the basic elements, such as computers, switches, and other devices, while edges represent the channels by which data is conveyed. Numerical numbers linked to the properties, linkages, and interactions in computer networks are known as topological indices or invariants. In this work, several topological complexity indices are constructed by employing paired trees up to the k-level. The results of this study have the potential to characterize and evaluate the topology of computer networks as well as chemical networks [15].

It has been successfully accomplished to evaluate the irregularity indices for oxide, hexagonal, silicate, and honeycomb networks. These results provide important new understandings of the behavior of various chemical and computer networks. Chemical and computer scientists can build their own top networks if they understand these ideas [16].

The research emphasizes the use of graph theory, a field of study that uses topological indices to understand the characteristics of different materials and networks without requiring empirical investigations. In order to obtain insight, this strategy entails creating mathematical formulae or equations for these substances and networks. The paper computes topological indices for m-polynomial square shift networks, a component present in many chemical compounds, using edge division [17].

This work investigates the derivation of topological indices and the mathematical representation of chemical structures. These indices are connected to certain strain energy, stability, and cutoff physiochemical properties of chemical substances. In this field of study, graph theory has shown to be quite beneficial. Studying the topology of certain networks has gained popularity recently. The present study extends the analysis of the authors to interconnection networks and draws logical results for the General Randi index R(G) in condensed valuable crystals and toroidal polyhex and octagonal networks, covering a wide range of conceivable values. In order to accommodate many processing centers with consistent processor-memory units, multiprocessor interconnection networks are frequently constructed [18].

The results of the study show that a supra-molecular catalytic system may be produced by combining a high number of catalytic sites with certain spatial configurations. This method works well because it speeds up reaction times by promoting the formation of pertinent advanced states. Many chemical processes that depend on multisite catalysis in network systems may benefit from the porous topologies of Metal-Organic Frameworks (MOFs) [19]. The report highlights how widely computers are used as information access tools, which has led to a surge in computer use across a range of businesses.

These days, it's difficult to identify any industry that hasn't been affected by computer applications—intelligence-related or not. Complex sets of legislation are even impacted by data systems and computational processing. Here, topological descriptors—numerical representations used to evaluate important information and understand the connections between the structure and characteristics of different materials—are essential to Quantitative Structure-Activity Relationship (QSAR) methodologies [22].

Computational and data systems play a crucial role in the development and implementation of complex legal systems, which frequently comprise elaborate webs of interrelated regulations or standards. Considering different computer paradigms, these regulatory networks are essential for controlling basic biological processes and directing analytical study. In order to control computational techniques in logical analysis and guarantee appropriate and correct outcomes, a comprehensive legal framework is necessary [23].

The study highlights how effective graphs are as modelling tools for expressing and illuminating a wide range of connections between intricate real-world problems. Graph theory may look at many different issues and challenges. By focusing on various graph theory problems and their implementation in software applications, this study demonstrated the significance of graph theory in network architecture. This page provides relevant information on graph theory and its applications in several domains, including operating systems, networks, databases, software engineering, etc., to assist software researchers. The study finds several practical applications of graph theory that are very relevant to the field of software development and computer programming [24].

In many computers programming applications, including networking, data mining, picture segmentation, and clustering, graph theory is essential. It makes data organization and network modelling easier. Resource reservation and allocation make use of fundamental ideas from graph theory, such as graph coverage. Resource network management, database configuration, and optimization difficulties are all handled via circuits and pathfinding methods. These applications encourage the creation of novel concepts and techniques that may be used to a variety of computer-related industries. The paper is split into two sections: the first explores the various computer applications where graph theory is useful, while the second gives an outline of how graph theory might be used in resource booking [25].

These domains have given rise to novel ideas in graph theory that tackle difficult graph theory issues. Significant progress is anticipated as long as graph theory and its applications continue to interact. The most useful use of graph theory in computer applications is in the creation of graph algorithms [26].

Topological indices offer a useful method for quantitatively identifying the essential characteristics of different systems and offer important insights on logarithmic structures [27].

3. Research Methodology

A methodical strategy is used in the research process. It starts by locating pertinent Benes networks to investigate. After that, these networks are represented as graphs utilizing graph theory ideas. To capture important features of Benes networks, specific Contraharmonic-Quadratic indices are created. Edge partitioning and formula-based indices are used in the procedure. To guarantee data accuracy, stringent validation techniques—such as the use of simulation programmes like Maple—are used.

This approach offers a systematic and trustworthy way to analyze Benes network behavior using Contraharmonic-Quadratic indices.

Objective: The objective of this work is to investigate and assess the Contraharmonic-Quadratic indices of recently proposed Benes network types. The study aims to investigate the features and attributes of these indices, evaluate their usefulness for network analysis, and determine their ability to accurately measure and capture important elements of Benes networks. The study's main goal is to improve our knowledge of these indicators' behavior, effectiveness, and suitability for use in the analysis of intricate network architectures.

Significance: This discovery is extremely significant since it can improve our knowledge of Benes networks, offer insightful information about complex network architectures, and give useful applications for network research and optimization across a range of industries.

Hypothesis: The following are the hypotheses of our research:

(i) The Benes network structure will influence how sensitively Contraharmonic-Quadratic indices react to perturbations.

(ii) Contraharmonic-quadratic indices will offer information on the information flow and connection patterns inside Benes networks.

(iii) There will be a correlation between Contraharmonic-Quadratic indices and other network measures and attributes.

(iv) It is possible to assess the overall performance, fault tolerance, and efficiency of Benes networks using Contraharmonic-quadratic indices.

3.1. Methods

Choosing a group of Benes networks that belong to the target class is the first stage. Then, these networks are represented using graph theory, which turns them into graphs with edges standing in for communication links and vertices for processing units or components. Specialized Contraharmonic-Quadratic indices created especially for Benes networks are introduced to collect pertinent network properties. These indices are used to analyze the topology of the network, specifically concentrating on edge classification. The acquired results are cross-checked against previously collected data to guarantee their efficacy. The study methodology is given and recorded in detail, emphasizing the usefulness and suitability of the Contraharmonic-Quadratic indices for different types of Benes networks.

4. Main Results

In [30, 31], the Quadratic-Contra harmonic indices (QCIs) and the Contra harmonic-quadratic indices (CQIs) respectively

$$QCI(G) = \sum \frac{du + dv}{\sqrt{2((du)^2 + (dv)^2)}} \quad (1)$$

$$CQI(G) = \sum \frac{\sqrt{2((du)^2 + (dv)^2)}}{du + dv} \quad (2)$$

In [32], V.R Kulli defined the geometric quadratic index and quadratic geometric index as

$$GQI(G) = \sum \frac{\sqrt{2((du)^2 . (dv)^2)}}{\sqrt{((du)^2 + (dv)^2)}} \quad (3)$$

$$QGI(G) = \sum \frac{\sqrt{((du)^2 + (dv)^2)}}{\sqrt{2((du)^2 . (dv)^2)}} \quad (4)$$

In [33], the arithmetic-contra harmonic index and the contra harmonic-arithmetic index
Defined by V.R Kulli is as

$$ACI(G) = \sum \frac{(du + dv)^2}{2((du)^2 + (dv)^2)} \quad (5)$$

$$CAI(G) = \sum \frac{2((du)^2 + (dv)^2)}{(du + dv)^2} \quad (6)$$

4.1. Cylindrical Representation of Benes Network

We connect the vertices of two types of cylindrical Benes networks in this section. We start by linking the vertices of the top row and the bottom row in a Benes network B(m). The network formed by this connection is referred to as a horizontal cylindrical Benes network ($HCB(m)$). The network is represented by $HCB(m)$. To illustrate this design, we provide an example of a three-dimensional horizontal cylindrical Benes network, or $HCB(3)$, in Figure 2. The labels assigned to the vertices in the network reveal their identities. Thus, the number of vertices in $HCB(m)$ is $|V(B(m))| = (2m+1)(2^m - 1)$ and no. of edges are $|E(B(m))| = 2m(2^{m+1} - 1)$.

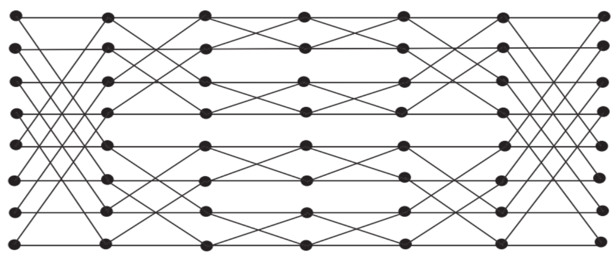

Figure 1 . 3-dimensional Benes Network

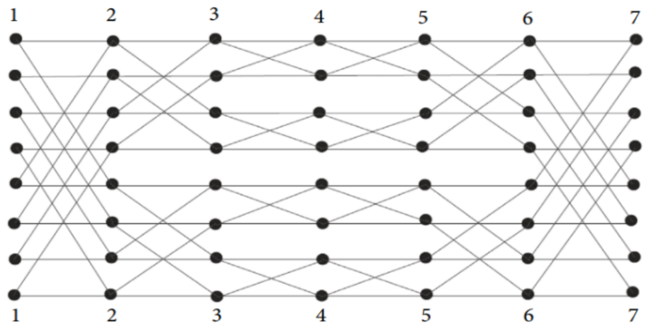

Figure 2 . 3-dimension horizontal cylindrical Benes network

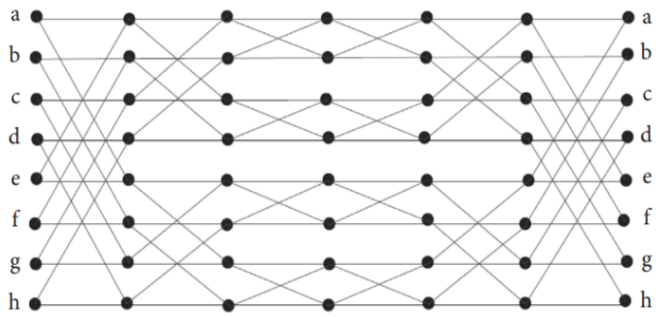

Figure 3 . 3-dimensional vertical cylindrical Benes network

Table 1. Partition of set of edge of $HCB(m)$ on the bases of degree of end vertices of every edge

$(\mathfrak{u}, \mathfrak{v})$	NUMBER OF EDGES
(2,6)	4
(3,4)	4
(3,6)	2
(4,2)	$2^{m+2} - 12$
(4,4)	$4(m-1)(2^m - 3)$
(4,6)	$8(m-1)$
(6,6)	$2(m-1)$

*Theorem 4.1.1***:** Let $\mathcal{G}$ be graph of horizontal cylindrical Benes network $HCB(\mathcal{G}(m))$ then

a) $CQI\left(HCB(\mathcal{G}(m))\right) = 10.36 + 2^{m+2}(m + 0.05) - 10m$

b) $QCI\left(HCB(\mathcal{G}(m))\right) = 0.11 + 2^m(4m - 3.06) - 2.16m$

c) $GQI\left(HCB(\mathcal{G}(m))\right) = 10.692 + 2^{m+2}(m - 1) - 2.308m$

d) $QGI\left(HCB(\mathcal{G}(m))\right) = 9.92 + 2^{m+2}(m + 0.11) - 1.68m$

e) $ACI\left(HCB(\mathcal{G}(m))\right) = 10.692 + 2^{m+2}(m - 1) - 2.308m$

f) $CAI\left(HCB(\mathcal{G}(m))\right) = 9.92 + 2^{m+2}(m + 0.11) - 1.68m$

Proof:

a) The CQI is given in equation (1)

$$\text{CQI} = \sum \frac{\sqrt{2((\mathfrak{u})^2+(\mathfrak{v})^2)}}{\mathfrak{u}+\mathfrak{v}} = \frac{\sqrt{2((2)^2+(6)^2)}}{2+6}(4) + \frac{\sqrt{2((3)^2+(4)^2)}}{3+4}(4) + \frac{\sqrt{2((3)^2+(6)^2)}}{3+6}(2) + \frac{\sqrt{2((4)^2+(2)^2)}}{4+2}(2^{2m} - 12) +$$

$$\frac{\sqrt{2((4)^2+(4)^2)}}{4+4}\left(4(m-1)(2^m - 3)\right) + \frac{\sqrt{2(4)^2+(6)^2)}}{4+6}\left(8(m-1)\right) + \frac{\sqrt{2(6)^2+(6)^2)}}{6+6}(2(m-1))$$

$= 10.36 + 2^{m+2}(m + 0.05) - 10m$

b). Now

$$QCI = \sum \frac{u+v}{\sqrt{2((u)^2+(v)^2)}} = \frac{2+6}{\sqrt{2((2)^2+(6)^2)}}(4) + \frac{3+4}{\sqrt{2((3)^2+(4)^2)}}(4) + \frac{3+6}{\sqrt{2((3)^2+(6)^2)}}(2) + \frac{4+2}{\sqrt{2((4)^2+(2)^2)}}(2^{2m} - 12) + \frac{4+4}{\sqrt{2((4)^2+(4)^2)}}\left(4(m-1)(2^m - 3)\right) + \frac{4+6}{\sqrt{2((4)^2+(6)^2)}}\left(8(m-1)\right) + \frac{6+6}{\sqrt{2((6)^2+(6)^2)}}\left(2(m-1)\right)$$

$= 0.11 + 2^m(4m - 3.06) - 2.16m.$

c). $$GQI = \sum\sqrt{\frac{2dudv}{du^2+dv^2}} = \sqrt{\frac{2(2)(6)}{2^2+6^2}}(4) + \sqrt{\frac{2(3)(4)}{3^2+4^2}}(4) + \sqrt{\frac{2(3)(6)}{3^2+6^2}}(2) + \sqrt{\frac{2(4)(2)}{4^2+2^2}}(2^{m+2} - 12) + \sqrt{\frac{2(4)(4)}{4^2+4^2}}\left(4(m-1)(2^m - 3)\right) + \sqrt{\frac{2(4)(6)}{4^2+6^2}}\left(8(m-1)\right) + \sqrt{\frac{2(6)(6)}{6^2+6^2}}(2(m-1) = 0.391 + 2^{m+2}(m - 0.106) - 2.134m$$

d) $$.QGI = \sum\sqrt{\frac{du^2+dv^2}{2dudv}} = \sqrt{\frac{2^2+6^2}{2(2)(6)^2}}(4) + \sqrt{\frac{3^2+4^2}{2(3)(4)^2}}(4) + \sqrt{\frac{3^2+6^2}{2(3)(6)^2}}(2) + \sqrt{\frac{4^2+2^2}{2(4)(2)^2}}(2^{m+2} - 12) + \sqrt{\frac{4^2+4^2}{2(4)(6)^2}}\left(4(m-1)(2^m - 3)\right) + \sqrt{\frac{4^2+6^2}{2(4)(6)^2}}\left(8(m-1)\right) + \sqrt{\frac{6^2+6^2}{2(6)(6)^2}}\left(2(m-1)\right) = -0.165 + 2^{m+2}(m + 0.11) - 1.674m$$

e). $$ACI = \sum\frac{(du+dv)^2}{2(du^2+dv^2)} = \frac{(2+6)^2}{2(2^2+6^2)}(4) + \frac{(3+4)^2}{2(3^2+4^2)}(4) + \frac{(3+6)^2}{2(3^2+6^2)}(2) + \frac{(4+2)^2}{2(4^2+2^2)}(2m^{m+2} - 12) + \frac{(4+4)^2}{2(4^2+4^2)}\left(4(m-1)(2^m - 3)\right) + \frac{(4+6)^2}{2(4^2+6^2)}\left(8(m-1)\right) + \frac{(6+6)^2}{2(6^2+6^2)}(2(m-1) = 10.692 + 2^{m+2}(m - 1) - 2.308m$$

f). $$CAI = \sum\frac{2(du^2+dv^2)}{(du+dv)^2} = \frac{2(2^2+6^2)}{(2+6)^2}(4) + \frac{2(3^2+4^2)}{(3+4)^2}(4) + \frac{2(3^2+6^2)}{(3+6)^2}(2) + \frac{2(4^2+2^2)}{(4+2)^2}(2^{m+2} - 12) + \frac{2(4^2+4^2)}{(4+4)^2}\left(4(m-1)(2^m - 3)\right) + \frac{2(4^2+6^2)}{(4+6)^2}\left(8(m-1)\right) + \frac{2(6^2+6^2)}{(6+6)^2}\left(2(m-1)\right) = 9.92 + 2^{m+2}(m + 0.11) - 1.68m$$

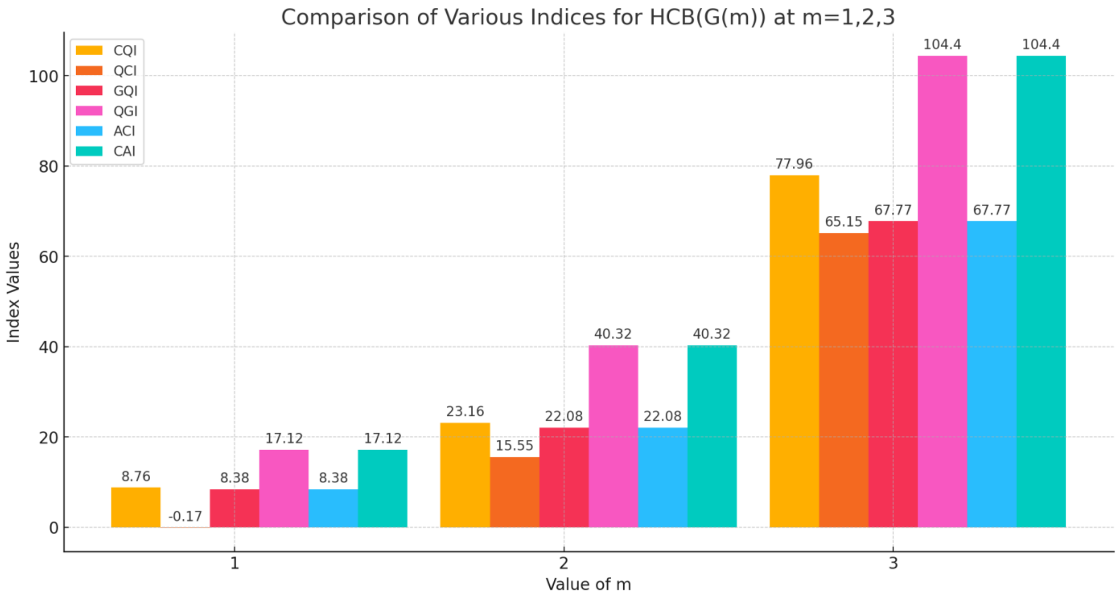

Figure 4. Graph 1: Comparison Bar Graph of Indices of $HCB(m)$ for $m = 1,2,3$

The bar graph for HCB(G(m)) exhibits distinct trends in the indices as m increases. The QCI and ACI start negatively at m=1 but step by step increase, underscoring their growing relevance with large network scales. In comparison, CQI, although beginning low, would not increase as sharply, indicating a steadier assessment of complexity and fine. GQI and CAI display the most tremendous growth, suggesting they are fantastically sensitive to modifications in m and may be important for evaluating community performance and adaptability at better scales. This contrast highlights how every index serves one-of-a-kind evaluative purposes relying on the community length and complexity.

Table 2. Partition of edge set of $HCB(m), m \geq 4$, built on degree sum of neighbor Partition of end vertices of each edge

$(\mathfrak{u}, \mathfrak{v})$	NUMBER OF EDGES
(8,12)	$2(2^{m+1}-1)$
(12,16)	$2(2^{m+1}-1)$
(16,16)	$2(2^{m+1}-1)(m-2)$

Theorem 4.1.2: Let $\mathcal{G}$ be graph of horizontal cylindrical Benes network $HCB(\mathcal{G}(m))$ $m \geq 4$ then

a) $CQI(HCB(\mathcal{G}(m)) = (2^{m+1}-1)(2m+0.05)$

b) $QCI(HCB(\mathcal{G}(m)) = (2^{m+1}-1)(2m-0.07)$

c) $GQI\left(HCB(\mathcal{G}(m))\right) = (2^{m+1}-1)(2m-0.12)$

d) $QGI\left(HCB\left(\mathcal{G}(m)\right)\right) = (2^{m+1}-1)(2m+0.12)$

e) $ACI\left(HCB(\mathcal{G}(m))\right) = 2(2^{m+1}-1)(2m-0.12)$

f) $CAI\left(HCB\left(\mathcal{G}(m)\right)\right) = 2(2^{m+1}-1)(2m+0.12)$

Proof:

a). As we know

$$CQI = \sum \frac{\sqrt{2((\mathfrak{u})^2+(\mathfrak{v})^2)}}{\mathfrak{u}+\mathfrak{v}} = \frac{\sqrt{2((8)^2+(12)^2)}}{8+12}\left(2(2^{m+1}-1)\right) + \frac{\sqrt{2((12)^2+(16)^2)}}{12+16}\left(2(2^{m+1}-1)\right) + \frac{\sqrt{2((16)^2+(16)^2)}}{16+16} \times \left(2(2^{m+1}-1)\right)(m-2) = (2^{m+1}-1)(2m+0.05)$$

b). $$QCI = \sum \frac{\mathfrak{u}+\mathfrak{v}}{\sqrt{2((\mathfrak{u})^2+(\mathfrak{v})^2)}} = \frac{8+12}{\sqrt{2((8)^2+(12)^2)}}\left(2(2^{m+1}-1)\right) + \frac{12+16}{\sqrt{2((12)^2+(16)^2)}}\left(2(2^{m+1}-1)\right) + \frac{16+16}{\sqrt{2((16)^2+(16)^2)}} \times \left(2(2^{m+1}-1)\right)(m-2) = (2^{m+1}-1)(2m-0.07).$$

c). $$GQI = \sum \sqrt{\frac{2dudv}{du^2+dv^2}} = \sqrt{\frac{2(8)(12)}{8^2+12^2}}(2(2^{m+1}-1)) + \sqrt{\frac{2(12)(16)}{12^2+16^2}}(2(2^{m+1}-1)) + \sqrt{\frac{2(16)(16)}{16^2+16^2}}(2(2^{m+1}-1)(m-2)) = (2^{m+1}-1)(2m-0.12)$$

d). $$QGI = \sum \sqrt{\frac{du^2+dv^2}{2dudv}} = \sqrt{\frac{8^2+12^2}{2(8)(12)^2}}\left(2(2^{m+1}-1)\right) + \sqrt{\frac{12^2+16^2}{2(12)(16)^2}}\left(2(2^{m+1}-1)\right) + \sqrt{\frac{16^2+16^2}{2(16)(16)^2}}\left(2(2^{m+1}-1) \times (m-2)\right) = (2^{m+1}-1)(2m+0.12)$$

e). $$ACI = \sum \frac{(du+dv)^2}{2(du^2+dv^2)} = \frac{(8+12)^2}{2(8^2+12^2)}\left(2(2^{m+1}-1)\right) + \frac{(12+16)^2}{2(12^2+16^2)}\left(2(2^{m+1}-1)\right) + \frac{(16+16)^2}{2(16^2+16^2)}(2(2^{m+1}-1)(m-2) = 2(2^{m+1}-1)(2m-0.12)$$

f). $$CAI = \sum \frac{2(du^2+dv^2)}{(du+dv)^2} = \frac{2(8^2+12^2)}{(8+12)^2}\left(2(2^{m+1}-1)\right) + \frac{2(12^2+16^2)}{(12+16)^2}\left(2(2^{m+1}-1)\right) + \frac{2(16^2+16^2)}{(16+16)^2}(2(2^{m+1}-1) \times (m-2)) = 2(2^{m+1}-1)(2m+0.12)$$

In the following comparison graph, ACI and CAI, those indices always perform the best across all values of m. This indicates that once changes for complexity are made, the ensuing price notably will increase, suggesting that these metrics may component in extra complexities or enhancements inside the network's shape that aren't accounted for inside the different indices. CQI and QGI, those indices are notably excessive but do not reach the stages of ACI and CAI. CQI considers factors that slightly decorate the complexity, while QGI consists of a advantageous adjustment, suggesting a barely extra positive evaluation of best boom. QCI and GQI, those indices display the lowest values, indicating a greater conservative estimation of first-class and increase, respectively, in the community shape. They subtract a small consistent factor from the bottom calculation, doubtlessly reflecting constraints or barriers within the network's design or overall performance.

Overall, the higher performance of ACI and CAI shows that those indices remember more distinct or substantial components of community complexity and version, which can be critical for programs requiring robust community evaluation and optimization. The decrease values of QCI and GQI may be beneficial for situations where a conservative estimate is essential, such as preliminary planning levels or when managing tremendously sensitive records or operations

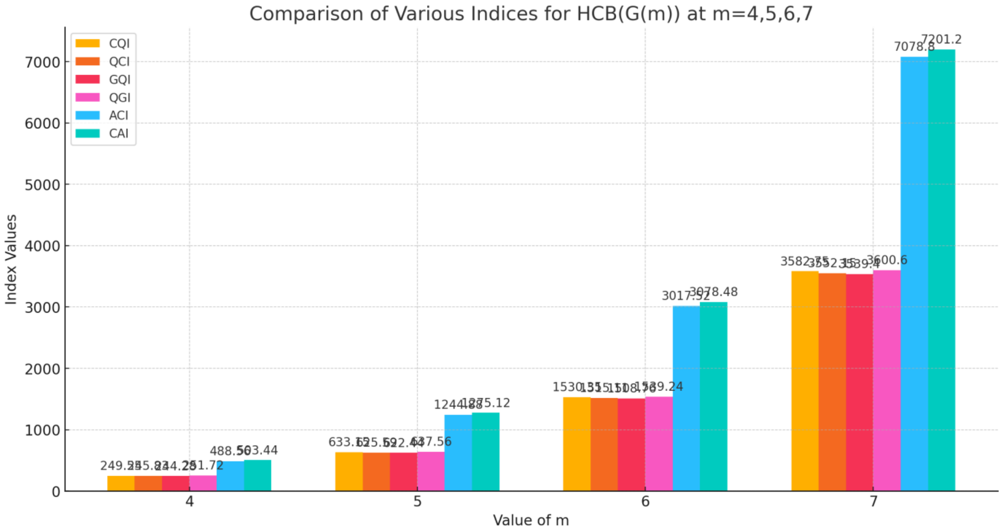

Figure 5. Graph 2: Comparison Bar Graph of Indices of $HCB(m)$ for $m = 4{,}5{,}6{,}7$

4.2. Toroidal Representation of Benes Network $TB(m)$

We may create a toroidal network called the Toroidal Benes network by calculating the number of vertices in the first row of VCB (m) and **comparing** it to the equivalent vertices of the last row. This network is represented by the symbol TB (m). The graph of the toroidal Benes net-work$TB(3)$ is displayed in Figure 4. Clearly, $|V(TB(m))| = (2m+1)(2^m - 1)$ and$|E(TB(m))| = 2m(2^{m+1} - 1)$.

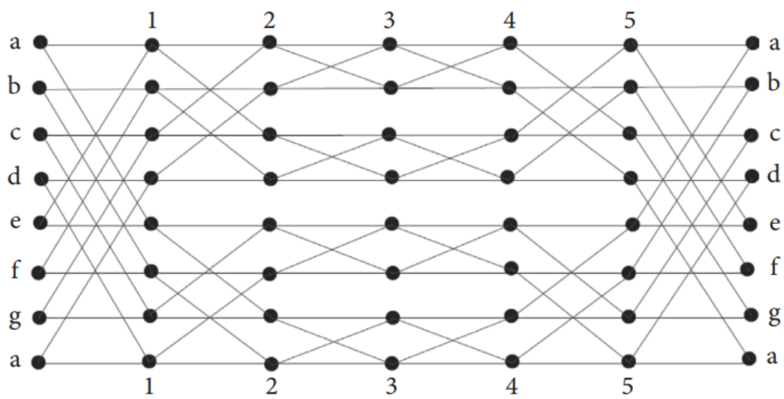

Figure 6. 3-dimensional toroidal Benes network

Table 3. Partition of Edge Set of $TB(m)$ Built on the Degree of End Vertices of Each Edge

$(\mathfrak{u}, \mathfrak{v})$	Number of Edges
(4,4)	$2m(2^{m+1} - 6)$
(4,6)	8m
(6,6)	2m

Theorem 4.2.1: Let $\mathcal{G}$ be the graph of toroidal cylindrical Benes network, $TB(\mathcal{G}(m))$ then

a) $CQI(TB(\mathcal{G}(m)) = m[2^{m+2} - 1.85]$

b) $QCI(TB(\mathcal{G}(m)) = m[2^{m+2} - 2.16]$

c) $ACI(TB(\mathcal{G}(m)) = 2(2^{m+2} - 2.308)$

d) $CAI(TB(\mathcal{G}(m)) = m(2^{m+2} - 1.68)$

e) $GQI(TB(\mathcal{G}(m)) = 4m(2^{m+2} + 0.4134)$

f) $QGI(TB(\mathcal{G}(m)) = m[0.25(2^{m+2}) - 9.9688]$

Proof:

a). The formula for the Contra harmonic quadratic index (CQI) is

$$\mathrm{CQI} = \sum \frac{\sqrt{2((\mathfrak{u})^2+(\mathfrak{v})^2)}}{\mathfrak{u}+\mathfrak{v}} = \frac{\sqrt{2((4)^2+(4)^2)}}{4+4}\left(2m(2^{m+1}-6)\right) + \frac{\sqrt{2((4)^2+(6)^2)}}{4+6}(8m) + \frac{\sqrt{2((6)^2+(6)^2)}}{6+6}(2m) \quad = m[2^{m+2} - 1.85]$$

b). the formula for the quadratic-Contra harmonic quadratic index (QCI) is

$$\mathrm{QCI} = \sum \frac{\mathfrak{u}+\mathfrak{v}}{\sqrt{2((\mathfrak{u})^2+(\mathfrak{v})^2)}} = \frac{4+4}{\sqrt{2((4)^2+(4)^2)}}\left(2m(2^{m+1}-6)\right) + \frac{4+6}{\sqrt{2((4)^2+(6)^2)}}(8m) + \frac{6+6}{\sqrt{2((6)^2+(6)^2)}}(2m) = m[2^{m+2} - 2.16]$$

c). Now for arithmetic Contra harmonic index,

$$ACI = \sum \frac{(du+dv)^2}{2(du^2+dv^2)} = \frac{(4+4)^2}{2(4^2+4^2)}\left(2m(2^{m+1}-6)\right) + \frac{(4+6)^2}{2(4^2+6^2)}(8m) + \frac{(6+6)^2}{2(6^2+6^2)}(2m)$$

$$= (2^{m+2} - 2)(m - 0.059)$$

d). The Contra harmonic quadratic index is

$$CAI = \sum \frac{2(du^2+dv^2)}{(du+dv)^2} = \frac{2(4^2+4^2)}{(4+4)^2}\left(2m(2^{m+1}-6)\right) + \frac{2(4^2+6^2)}{(4+6)^2}(8m) + \frac{2(6^2+6^2)}{(6+6)^2}(2m)$$

$$= m(2^{m+2} - 1.68)$$

e). The geometric quadratic index is following $GQI = \sum\sqrt{\frac{2du^2dv^2}{du^2+dv^2}} = \sqrt{\frac{2(4^2)(4^2)}{4^2+4^2}}\left(2m(2^{m+1}-6)\right) + \sqrt{\frac{2(4^2)(6^2)}{4^2+6^2}}(8m) + \sqrt{\frac{2(6^2)(6^2)}{6^2+6^2}}(2m) = 4m(2^{m+2} + 0.4134)$

f). The quadratic geometric index is

$$QGI = \sum\sqrt{\frac{du^2+dv^2}{2du^2dv^2}} = \sqrt{\frac{4^2+4^2}{2(4^2)(4^2)}}\left(2m(2^{m+1}-6)\right) + \sqrt{\frac{4^2+6^2}{2(4^2)(6^2)}}(8m) + \sqrt{\frac{6^2+6^2}{2(6^2)(6^2)}}(2m) = m[0.25(2^{m+2}) - 9.9688]$$

In the following comparison graph, GQI, this index considerably outperforms the others for all values of m. The GQI method incorporates a strong growth component, scaled with the aid of 4m, and provides a positive constant, which significantly will increase with m. This indicates that the GQI is probably specifically touchy to aspects of the network's shape that beautify its ability for handling complex operations or expansions, making it important for situations wherein increase capability is a key aspect. The ACI does no longer scale with m, which results in a regular calculation throughout special values of m. This index gives a steady measure of complexity that might be useful for standardized assessments throughout diverse configurations or sizes of the community. The QGI increases at a decrease rate compared to different indices, indicating a conservative estimate of fine increase inside the network. This index might be essential in conservative designs or opinions in which overestimation of functionality ought to cause inefficiencies or failures. CQI, QCI , and CAI, these indices scale linearly with m and regulate barely for various factors. They provide a mild assessment of the community's skills, appropriate for general critiques however much less impactful while excessive precision or high sensitivity to modifications is required. Overall, the better price of the GQI throughout all values of m highlights its ability utility in packages requiring an assessment of growth skills. In assessment, the ACI's consistency might be useful for programs that need a stable complexity metric irrespective of the network length or configuration. The slight increases in CQI, QCI, and CAI suggest their applicability in routine assessments where intense elements are much less in all likelihood to impact the overall evaluation.

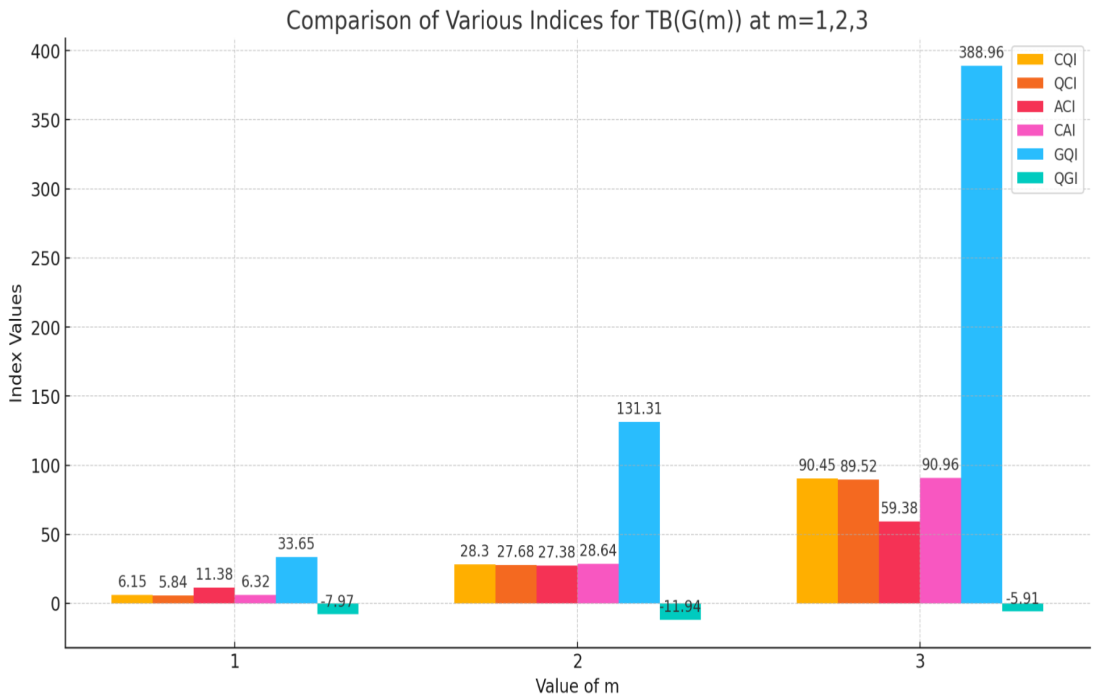

Figure 7. Graph 3: Comparison Bar Graph of Indices of $TB(m)$ for $m = 1,2,3$

4.3. Representation of Butterfly Network $BF(m)$

The butterfly network is among the most important and widely utilized degree-based networks. It is made up of butterfly-like motifs. The set of vertices V in an m-dimensional butterfly network comprises pairs (i, j), where i is an m-bit binary value that represents the row of the node and j indicates the level or stage of nodes (ranging from 0 to m). A walking cycle is used to link two nodes. This means that a node (i, j) is connected to two other nodes: a node $(i, j + 1)$ and a node $(m, j + 1)$, where m is the result of flipping the jth bit and reflects the butterfly pattern. The edges in these networks are not directed. $BF(m)$ represents an m-dimensional butterfly network with $2^m(r + 1)$ vertices and $m2^{m+1}$ are edges. Figure 5 shows a three-dimensional butterfly network,$BF(3)$.

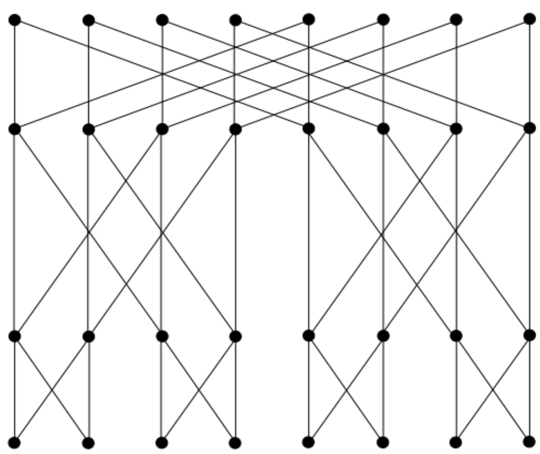

Figure 8. a 3-dimensional Butterfly Network

Table 4. Partition of edge set of BF(3) built on degree sum of neighbor Partition of end vertices of each edge

$(\mathfrak{u}, \mathfrak{v})$	NUMBER OF EDGES
(2,4)	2^{M+2}
(12,16)	$2^{M+2}(M-2)$

Theorem 4.3.1: Let $\mathcal{G}$ be the graph of toroidal cylindrical Benes network, $TB(\mathcal{G}(m))$ then

a) $CQI\,(BF(m)) = 2^{m+2}(9.6974m - 18.3408)$

b) $QCI(BF(m)) = 2^{m+2}[0.1031m + 0.7424]$

c) $QGI(BF(m)) = 2^{m+2}[0.0736m + 0.248]$

d) $ACI(BF(m)) = 2^{m+2}[0.0003m - 0.8994]$

e) $CIA(BF(m)) = 2^{m+2}[28.5714m - 56.5873]$

f) $GQI(BF(m)) = 2^{m+2}[13.5764m - 24.623]$

Proof:

a). As we know

$$\mathrm{CQI} = \sum \frac{\sqrt{2((\mathfrak{u})^2+(\mathfrak{v})^2)}}{\mathfrak{u}+\mathfrak{v}} = \frac{\sqrt{2((2)^2+(4)^2)}}{2+4}(2^{m+2}) + \frac{\sqrt{2((12)^2+(16)^2)}}{12+16}(2^{m+2}(m-2)) = 2^{m+2}(9.6974m - 18.3408)$$

b). For QCI

$$\mathrm{QCI} = \sum \frac{\mathfrak{u}+\mathfrak{v}}{\sqrt{2((\mathfrak{u})^2+(\mathfrak{v})^2)}} = \frac{2+4}{\sqrt{2((2)^2+(4)^2)}}(2^{m+2}) + \frac{12+16}{\sqrt{2((12)^2+(16)^2)}}(2^{m+2}(m-2)) = 2^{m+2}[0.1031m + 0.7424]$$

c). For QGI

$$QGI = \sum \sqrt{\frac{du^2+dv^2}{2du^2dv^2}} = \sqrt{\frac{2^2+4^2}{2(2^2)(4^2)}}(2^{m+2}) + \sqrt{\frac{12^2+16^2}{2(12^2)(16^2)}}(2^{m+2}(m-2)) = 2^{m+2}[0.0736m + 0.248]$$

d). For ACI

$$ACI = \sum \frac{(du+dv)^2}{2(du^2+dv^2)} = \frac{(2+4)^2}{2(2^2+4^2)}(2^{m+1}) + \frac{(12+16)^2}{2(12^2+16^2)}2^{m+2}(m-2) = 2^{m+2}[0.0003m - 0.8994]$$

e). For CAI

$$CAI = \sum \frac{2(du^2+dv^2)}{(du+dv)^2} = \frac{2(2^2+4^2)}{(4+4)^2}(2^{m+1}) + \frac{2(12^2+16^2)}{(12+16)^2}(2^{m+2}(m-2)) = 2^{m+2}[28.5714m - 56.5873]$$

f). For GQI

$$GQI = \sum \sqrt{\frac{2du^2dv^2}{du^2+dv^2}} = \sqrt{\frac{2(2^2)(4^2)}{2^2+4^2}}(2^{m+1}) + \sqrt{\frac{2(12^2)(16^2)}{12^2+16^2}}(2^{m+2}(m-2)) = 2^{m+2}[13.5764m - 24.623]$$

In the evaluation of indices for the toroidal cylindrical Benes network $BF(m)$, the CIA indicates the very best values, indicating a enormous have an effect on of complexity adjustments on network overall performance. The GQI also scales robustly with m highlighting its relevance in scenarios requiring tests of growth ability. The CQI, though starting decrease, grows exponentially, suggesting its usefulness in environments where network complexity and pleasant are paramount. Meanwhile, the QCI and QGI showcase slight increase, suitable for balanced reviews. The ACI remains drastically low, doubtlessly due to its components's layout, which can also simplest be effective below particular, excessive-scale situations. This shows various software of each index primarily based at the community's complexity and boom characteristics.

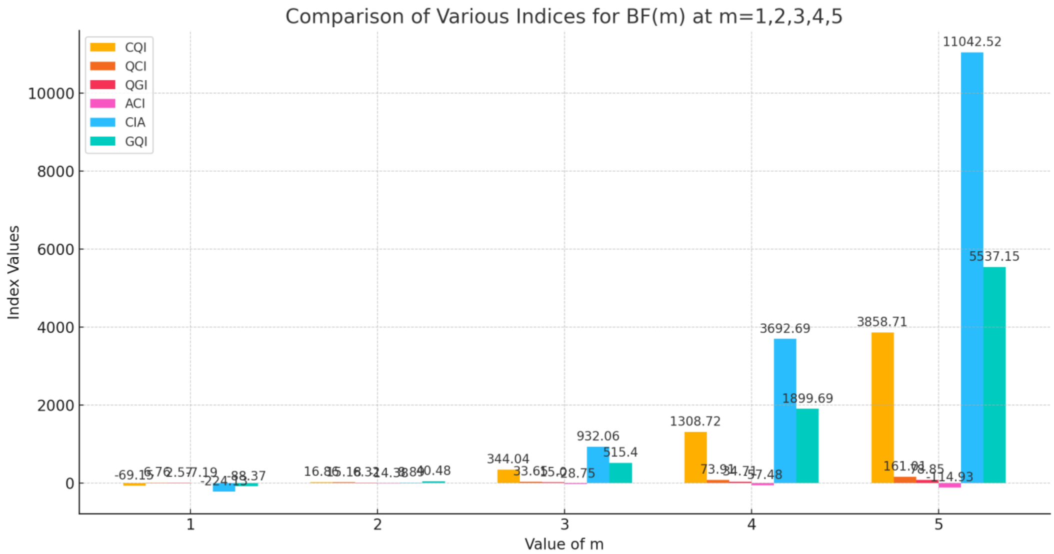

Figure 9. Graph 4, Comparison Bar Graph of Indices of $BF(m)$ for $m = 1,2,3,4,5$,

5. Conclusion

The obtained indices shed important light on the structural properties of cylindrical Benes networks and Butter Fly that are vertical and horizontal. These findings might improve performance and usefulness in a number of areas, including as boosting data center operations and communication effectiveness. The results of the study advance our knowledge of network behavior and the useful applications of Benes networks and butterfly network. The study's main finding emphasizes how important topological indices are for characterizing complex networks, such as Benes networks. These calculated indices are vital resources for understanding and enhancing network performance, especially in applications pertaining to digital signal processing, telephony, data centers, parallel computing, and Network-on-Chip architecture. It is anticipated that these discoveries would prove beneficial for the development and use of Benes networks in many practical scenarios.

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